

Tiling a Chessboard: A Problem Adapted for the Virtual Math Circle



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Outline

- Tucson Math Circle
- Anatomy of Miro (free collaborative whiteboard)
- Tiling Activities
- (Brief) Anatomy of GeoGebra
- Variations

Tucson Math Circle

Remote online format

- Started Fall 2020 Christina Durón
Nicole Sullivant
Doug Pfeffer
- Zoom and Miro.com

Tucson Math Circle

Remote online format

- Started **Fall 2020**
Christina Durón
Nicole Sullivant
Doug Pfeffer
- Zoom and Miro.com
- Two student groups: **Junior (K-6)** and Senior (7-12)
Christina Durón
John Peca-Medlin
Bruce Bayly
Megan Lewis
- **Our goal:** To have students explore and enjoy topics in mathematics in a way that is not part of the standard K-12 curriculum

Tucson Math Circle

Challenges

1. Wide range of ages/grades
 - Cannot expect learning/recall of technical details
2. Attendance is flexible
 - Students may miss weeks
3. Student fatigue during the scheduled meetings
 - Wednesday late afternoon
 - One hour per meeting
 - Biweekly meetings

Our Solutions

1. Incorporate warm-ups
2. Focus on demonstrations
3. Rely on group discussions and interactive activities

Anatomy of Miro

tinyurl.com/JMM-Tiling-Problem



Anatomy of Miro

tinyurl.com/JMM-Tiling-Problem



miro Dominoes ⚙️ ⬆️ 🔍

All frames Smart Meeting ×

▶ Present ...

Sign-In Sheet

Warm-Up 1

2 Warm-Up 1 ...

Warm-Up 2

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📄

Warm-Up 1

Warm-Up 1

Directions: Start with an uncovered chessboard. Can you cover the board with dominoes by following these rules?

RULE 1: Each domino should cover two squares of the board.

RULE 2: Dominoes are not allowed to overlap one another.

Follow-Up 1: If your answer is yes...can you find two or three *different* ways of covering the board?

Follow-Up 2: For each of these coverings, how many dominoes did you use? Why does that number make sense?

Name

Name

Name

Name

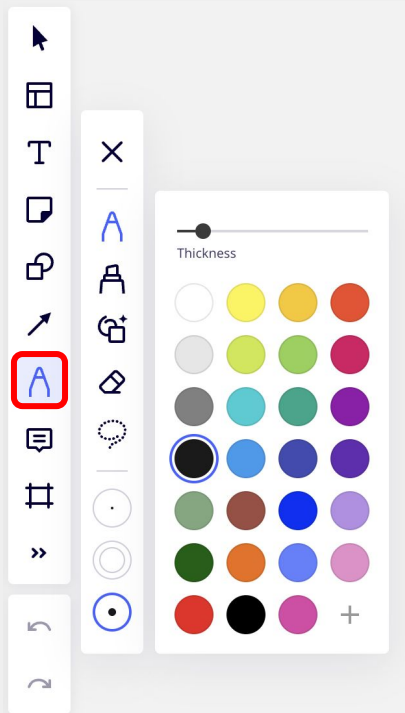
Name

Name

Name

Name

Anatomy of Miro

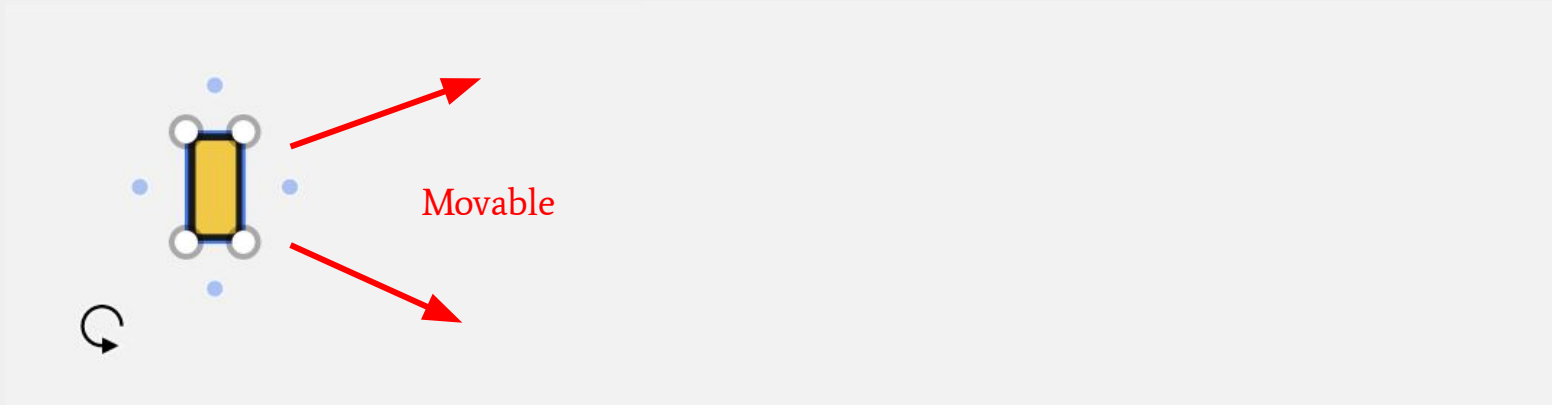


Student A

$8 \times 8 = 64$
 $64 \div 2 = 32$
32

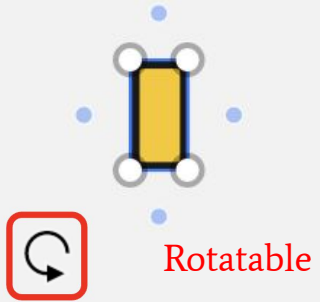
Anatomy of Miro

tinyurl.com/JMM-Tiling-Problem



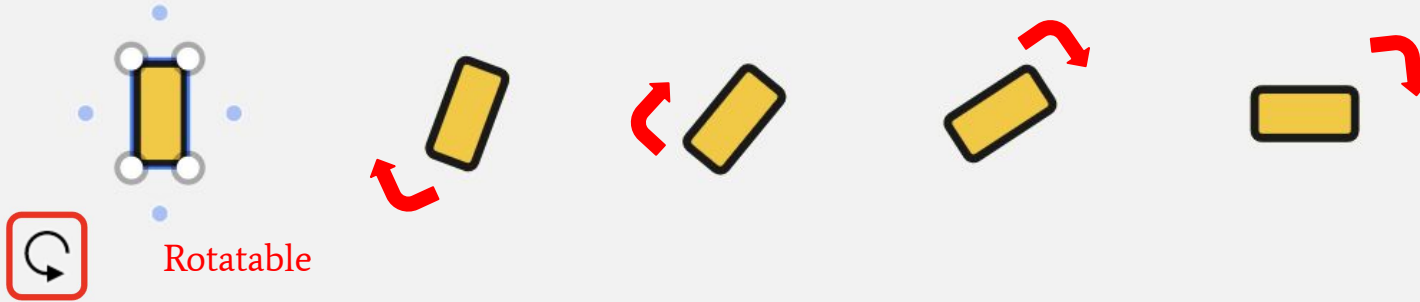
Anatomy of Miro

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tinyurl.com/JMM-Tiling-Problem



Anatomy of Miro

tinyurl.com/JMM-Tiling-Problem

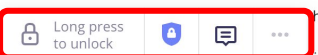



Directions: Start with an uncovered chessboard. Take one of the "off limits" markers (X) and use it to cover two spaces.


Following Rule 1, Rule 2, and Rule 3, which boards can be covered?

Which boards cannot be covered?

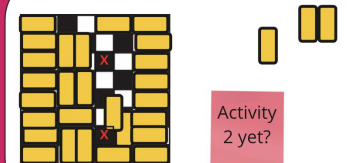
RULE 1: Each domino should cover two squares of the board.

RULE 2:  her.

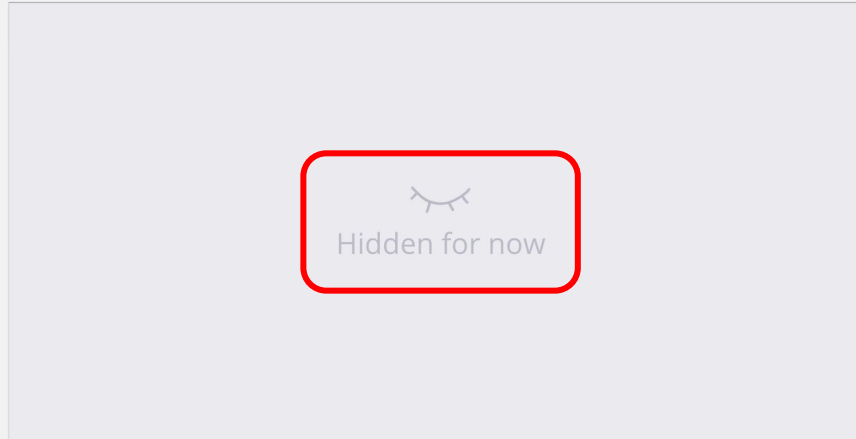
RULE 3:  "off limits" marker.



Student A



Activity 2



Day 1 Purposes

Dominoes on 8x8 chess boards

Warm-Up 1	Warm-Up 2	Activity 1
<ul style="list-style-type: none">● Miro functionality● Reviewing rules (<i>uniqueness</i>)	<ul style="list-style-type: none">● New rule with 1 “X” (<i>divisibility</i>)● Miro functionality	<ul style="list-style-type: none">● Generalizations with 2 “X”s

Day 1

- Miro functionality ✓
- Rules 1 - 2 ✓

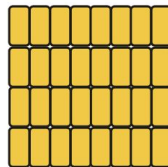
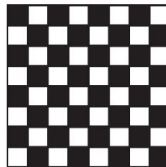
Day 1

- Warm-Up 1
- Warm-Up 2
- Activity 1

Directions: Start with an uncovered chessboard. Can you cover the board with dominoes by following these rules?

RULE 1: Each domino should cover two squares of the board.

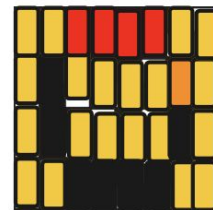
RULE 2: Dominoes are not allowed to overlap one another.



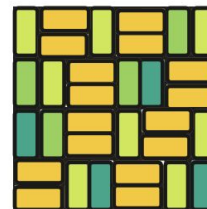
Follow-Up 1: If your answer is yes...can you find two or three *different* ways of covering the board?

Follow-Up 2: For each of these coverings, how many dominoes did you use? Why does that number make sense?

Student A



Student B



Day 1

- Rule 3 ✓
- Miro functionality ✓

Day 1

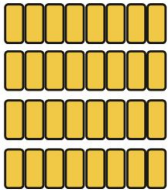
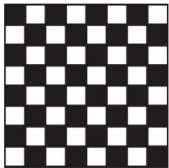
- Warm-Up 1
- Warm-Up 2
- Activity 1

Directions: Start with an uncovered chessboard. Take one of the "off limits" markers (X) and use it to cover the top right corner. Can you cover this board by following these rules?

RULE 1: Each domino should cover two squares of the board.

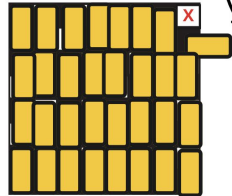
RULE 2: Dominoes are not allowed to overlap one another.

RULE 3 (new!): Dominoes are not allowed to cover the "off limits" marker.



Follow-Up: Think about your answer to Warm-Up #2. Why cannot you cover the board when one tile is off limits?

Student A



$b4 - 1 = b3$
 $b3 \div 2 = 3$

Day 1

- Generalizations

Day 1

- Warm-Up 1
- Warm-Up 2
- Activity 1

Directions: Start with an uncovered chessboard. Take one of the "off limits" markers (X) and use it to cover two spaces.

Following Rule 1, Rule 2, and Rule 3, which boards can be covered?

Which boards cannot be covered?

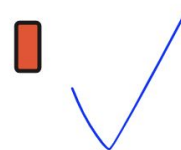
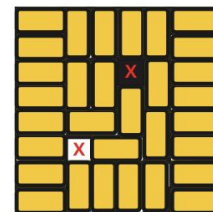
RULE 1: Each domino should cover two squares of the board.

RULE 2: Dominoes are not allowed to overlap one another.

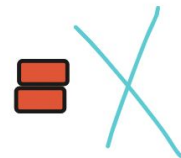
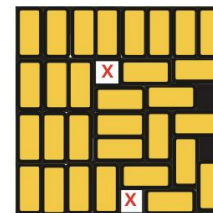
RULE 3: Dominoes are not allowed to cover the "off limits" marker.

Follow-Up: Do the boards that cannot be covered have anything in common? Explain why the pattern makes sense.

Student C



Student C



Day 1

- Generalizations ✓

Day 1

- Warm-Up 1
- Warm-Up 2
- Activity 1

Directions: Start with an uncovered chessboard. Take one of the "off limits" markers (X) and use it to cover two spaces.

Following Rule 1, Rule 2, and Rule 3, which boards can be covered?

Which boards cannot be covered?

RULE 1: Each domino should cover two squares of the board.

RULE 2: Dominoes are not allowed to overlap one another.

RULE 3: Dominoes are not allowed to cover the "off limits" marker.

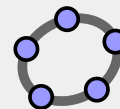
Follow-Up: Do the boards that cannot be covered have anything in common? Explain why the pattern makes sense.

Student B

Student B

Day 1

tinyurl.com/GeoGebra-Act1



Day 1

- Warm-Up 1
- Warm-Up 2
- **Activity 1**

Directions: Start with an uncovered chessboard. Take one of the "off limits" markers (X) and use it to cover two spaces.

Following Rule 1, Rule 2, and Rule 3, which boards can be covered?

Which boards cannot be covered?

RULE 1: Each domino should cover two squares of the board.

RULE 2: Dominoes are not allowed to overlap one another.

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Follow-Up: Do the boards that cannot be covered have anything in common? Explain why the pattern makes sense.

www.geogebra.org

Dominoes on a Chessboard

Dominoes on a Chessboard

Day 2 Purposes

Trominoes

Warm-Up 3	Activity 2	Activity 3
<ul style="list-style-type: none">● Miro functionality● Reviewing rules (<i>divisibility</i>)● Review Day 1	<ul style="list-style-type: none">● Rules with 1 “X” with 4x4 board (<i>base case</i>)	<ul style="list-style-type: none">● Rules with 1 “X” with 8x8 board (<i>induction</i>)● GeoGebra functionality

Day 2

Day 2

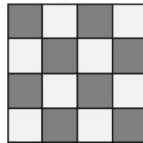
- Warm-Up 3
- Activity 2
- Activity 3

Directions: Start with an uncovered 4x4 board. Can you explain why it is *impossible* to cover the board with L-shaped “trominoes” if you must follow these rules?

RULE 1: Each tromino should cover three squares of the board.

RULE 2: Trominoes are not allowed to overlap.

Hint: How many spaces would be covered by 1 tromino? 2 trominos? 3 trominos? 5 trominos?



Day 2

- Rules 1 - 3 ✓

Day 2

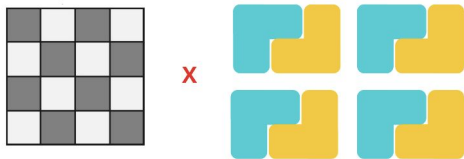
- Warm-Up 3
- Activity 2
- Activity 3

Directions: Start with an uncovered 4x4 board. Take one of the "off limits" markers (X) and use it to cover one space. Try to cover the board by following these rules:

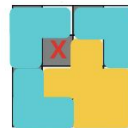
RULE 1: Each tromino should cover three squares of the board.

RULE 2: Trominoes are not allowed to overlap.

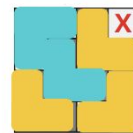
RULE 3 (new!): Trominoes are not allowed to cover the "off limits" marker.



Student A



Student B



Student C



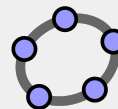
Day 2

- Rules 1 - 3 ✓

Day 2

- Warm-Up 3
- Activity 2
- Activity 3

tinyurl.com/GeoGebra-Act2

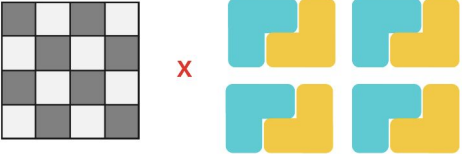


Directions: Start with an uncovered 4x4 board. Take one of the "off limits" markers (X) and use it to cover one space. Try to cover the board by following these rules:

RULE 1: Each tromino should cover three squares of the board.

RULE 2: Trominoes are not allowed to overlap.

RULE 3 (new!): Trominoes are not allowed to cover the "off limits" marker.



Chessboard



Trominoes



www.geogebra.org

Trominoes on a 4x4 Chessboard

Trominoes on a 4x4 Chessboard

Day 2

- Generalizations X

Day 2

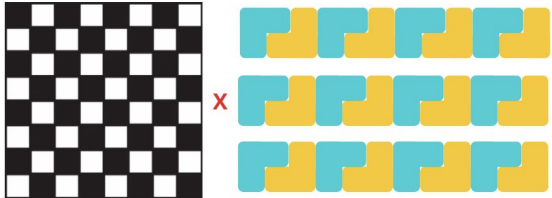
- Warm-Up 3
- Activity 2
- Activity 3

Directions: Start with an uncovered 8x8 board. Take one of the "off limits" markers (X) and use it to cover one space. Try to cover the board by following these rules:

RULE 1: Each tromino should cover three squares of the board.

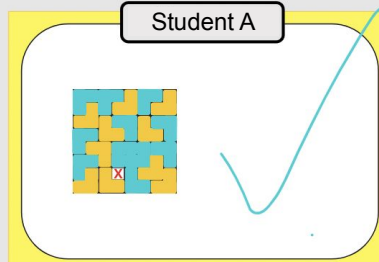
RULE 2: Trominoes are not allowed to overlap.

RULE 3 (new!): Trominoes are not allowed to cover the "off limits" marker.

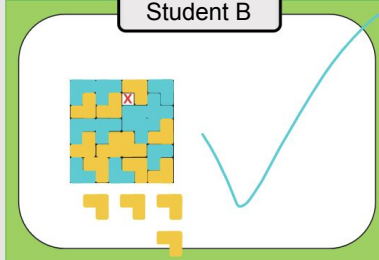


Follow-Up: Is there some "Divide and Conquer" strategy that we could use to make this task easier to complete?

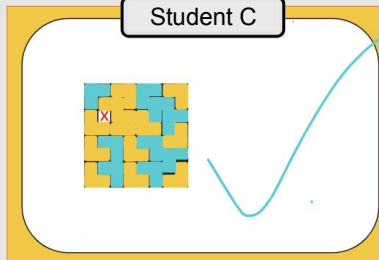
Student A



Student B

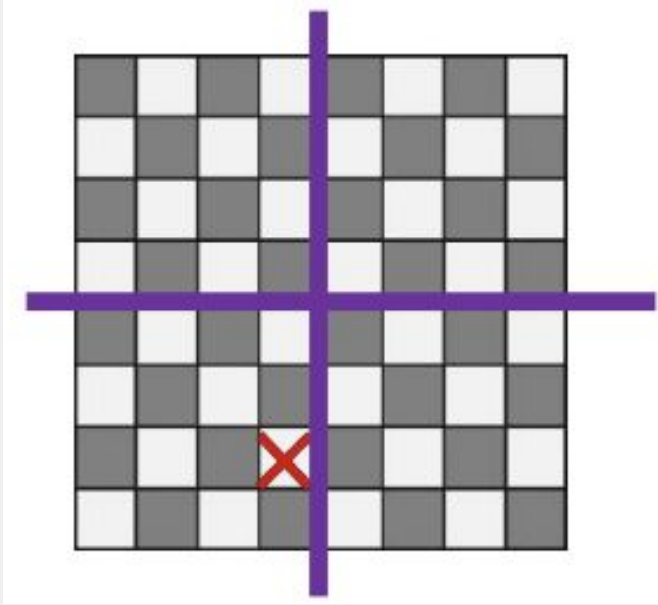


Student C

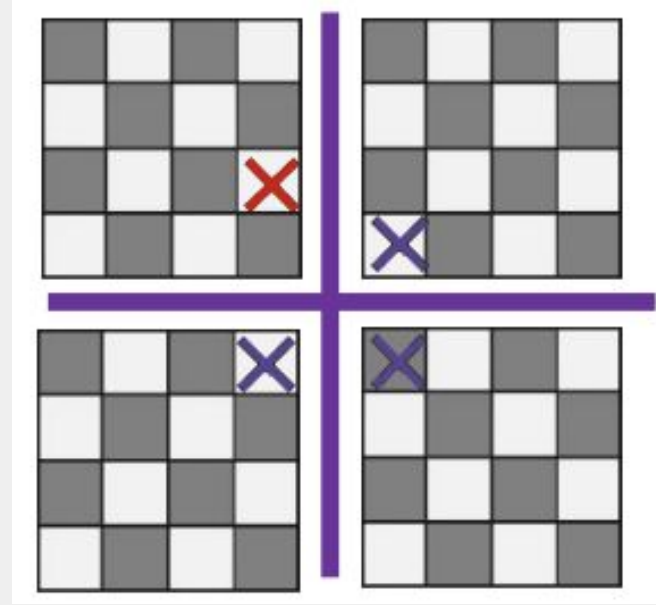


Divide and Conquer Strategy

Step 1:

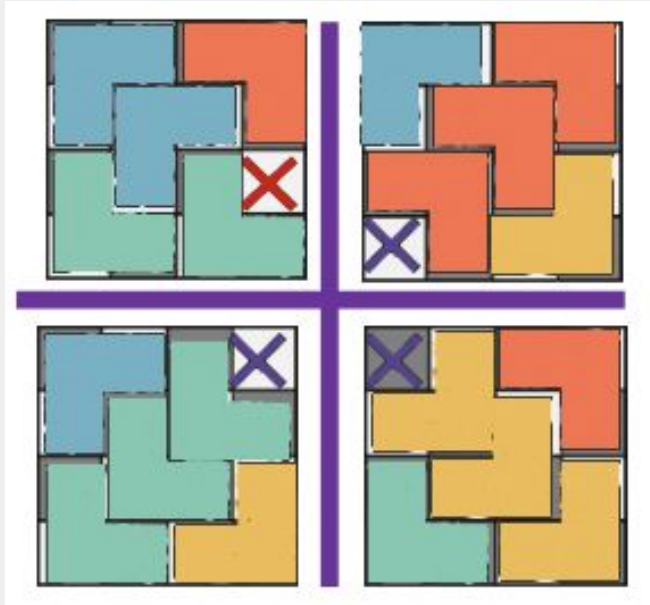


Step 2:

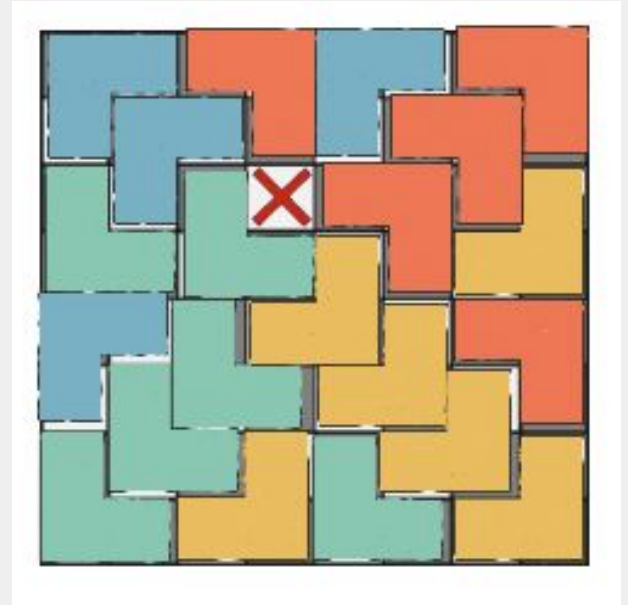


Divide and Conquer Strategy

Step 3:



Step 4:

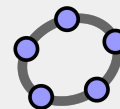


Day 2

Day 2

- Warm-Up 3
- Activity 2
- Activity 3

tinycloud.com/GeoGebra-Act3

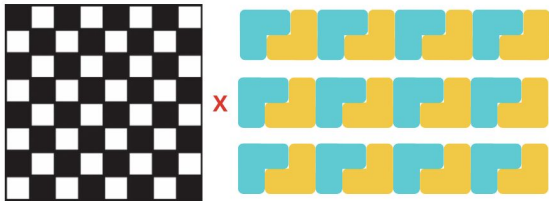


Directions: Start with an uncovered 8x8 board. Take one of the “off limits” markers (X) and use it to cover one space. Try to cover the board by following these rules:

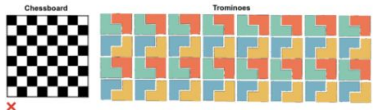
RULE 1: Each tromino should cover three squares of the board.

RULE 2: Trominoes are not allowed to overlap.

RULE 3 (new!): Trominoes are not allowed to cover the “off limits” marker.



Follow-Up: Is there some “Divide and Conquer” strategy that we could use to make this task easier to complete?



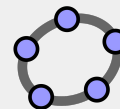
www.geogebra.org

Trominoes on an 8x8 Chessboard

Trominoes on an 8x8 Chessboard

Day 2

tinyurl.com/GeoGebra-Act3-4x



- GeoGebra functionality ✓

Day 2

- Warm-Up 3
- Activity 2
- Activity 3 - 4x

The screenshot displays a GeoGebra activity interface. At the top, the browser address bar shows the URL geogebra.org/m/ttswgfga. The page title is "Trominoes on an 8x8 Chessboard with 4 X's" by author Christina Durón. Below the title, a question asks: "Can you arrange the red, orange, blue, and green trominoes to cover the 8x8 chessboard, without any trominoes overlapping?". The main content area is divided into two sections: "Chessboard" on the left, which shows an 8x8 grid with four red 'X' marks, and "Trominoes" on the right, which shows a collection of red, orange, blue, and green L-shaped trominoes. A sidebar on the right side of the screen shows a video conference interface with two participants: John Peca-Medlin and Christina Duron.

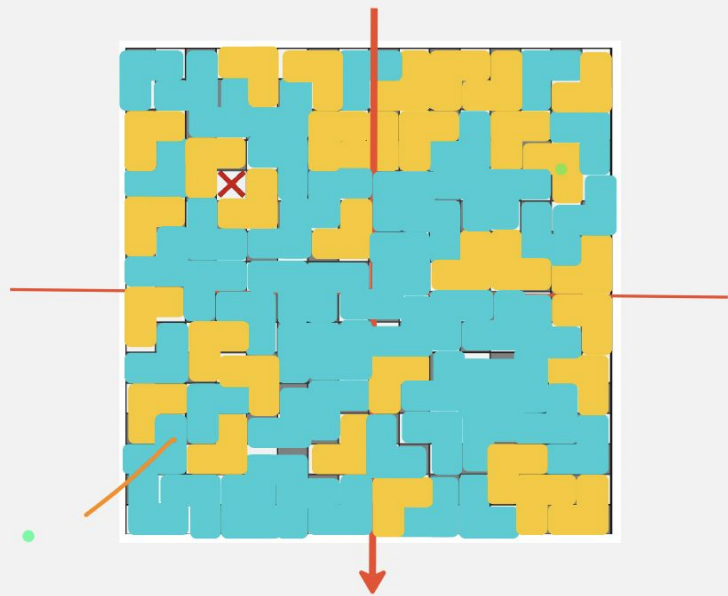
Day 2

Day 2

- Warm-Up 3
- Activity 2
- Activity 3
- Extension

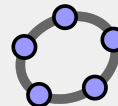
Challenge activity: Start with an uncovered 16x16 board. Take one of the “off limits” markers and use it to cover one space. Try to cover the board by following Rule 1, Rule 2, and Rule 3.

Hint: use the “Divide and Conquer” strategy!



Anatomy of GeoGebra

tinyurl.com/GeoGebra-Act3-4x



The screenshot displays the GeoGebra software interface. On the left, the Algebra window lists various objects: $A = (-54.68, -21.7)$, $B = (18.43, -21.7)$, $C = (27.05, 15.1)$, $D = (44.72, 15.1)$, $E = (26.57, 33.5)$, $F = (44.97, 33.5)$, $G = (36.25, 34)$, $H = (53.92, 34)$, $I = (36.49, 14.2)$, $J = (54.41, 14.74)$, $A_1 = (-47.39, -57.04)$, $B_1 = (25.72, -57.04)$, $C_1 = (27.17, -21.94)$, $D_1 = (44.85, -21.94)$, $E_1 = (26.69, -3.54)$, $F_1 = (45.09, -3.54)$, $G_1 = (36.37, -3.06)$, $H_1 = (54.04, -3.06)$, $I_1 = (36.61, -22.67)$, $J_1 = (54.53, -22.67)$, $A_2 = (-18.58, -21.7)$, $B_2 = (54.53, -21.7)$, $C_2 = (56.22, 15.1)$, $D_2 = (73.9, 15.1)$, $E_2 = (55.74, 33.5)$, $F_2 = (74.14, 33.5)$, $G_2 = (65.42, 33.98)$, and $H_2 = (83.1, 33.98)$. A menu is open over the algebra list, showing options: Web page (html) ..., Graphics View as Picture (png, eps) ..., Graphics View as Animated GIF ..., Graphics View to Clipboard, Graphics View as PSTricks ..., Graphics View as PGF/TikZ ..., Graphics View as Asymptote ..., Collada..., and Collada (html).... Below the menu, a 4x4 grid of black and white squares is visible, with four red 'X' marks below it. The main Graphics View displays a 10x4 grid of colorful trominoes (red, green, blue, yellow) under the title "Trominoes". The interface includes a toolbar at the top, a File menu, and an Algebra window on the left.

Variations

- **Nicole Sullivant** (Grades 7-8)

Domino Covering Challenge

Question: What conditions determine if these oddly-shaped boards can be covered by dominos?

Can you arrange the yellow and orange dominos in a way that covers the shaped boards by following these rules?  

Rule 1: Each domino should cover 2 squares
Rule 2: Dominoes should not overlap

Question: What conditions determine if these oddly-shaped boards can be covered by dominos?



Domino Covering: Proofs

Prove the following propositions:

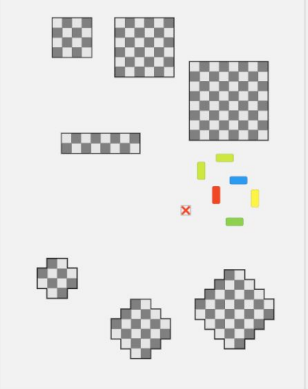
1. If a rectangular board has an even number of squares, then it must have an equal number of light squares and dark squares.
2. It is always possible to cut an L-shaped board into two rectangular regions, where one of the region has an even number of squares.
3. If an L-shaped board has an even number of squares, then it must have an equal number of light squares and dark squares.



Variations

- Ibrahim Fatkullin (Grades 9-12)

TEMPLATES



Take a board which consists of square cells (it does not need to be rectangular) and cover it with 2x1 dominoes. If you can cover the entire board, we call it a tiling.

For each of the boards below, determine if it can be tiled or not. If yes, demonstrate one such tiling. If not, show that it does not exist.

It is not hard to block off a square so that we cannot tile the remaining part of the board, but can you place 4 crosses on a 4x4 board (covering 2 light and two dark squares) so that no square is blocked off and yet the rest of the board cannot be tiled?

What is the minimal number of crosses we need to place on a 4x4 board so that we cover the same number of light and dark squares, do not block off any part of the board, and yet cannot tile the rest of the board?

IDEA! BLOCK OFF A SQUARE LIKE THIS

Introduction to Domino tilings

How many ways are there to tile these 2x2 boards? Construct all possible tilings for the first two boards. Do you see a pattern, can you prove it?

These boards are called **Anti-Diamonds**. Tile them as well. Can you guess the number of all possible tilings for them?

Dominoes of just one color are **Anti**. Let's use four colors (depending on how the domino pieces is aligned with respect to the underlying checkerboard pattern).

Work together to construct all possible completions of this tiling!

Counting tilings

How many ways are there to tile these 2x2 boards? Construct all possible tilings for the first two boards. Do you see a pattern, can you prove it?

These boards are called **Anti-Diamonds**. Tile them as well. Can you guess the number of all possible tilings for them?

Dominoes of just one color are **Anti**. Let's use four colors (depending on how the domino pieces is aligned with respect to the underlying checkerboard pattern).

Work together to construct all possible completions of this tiling!

Tiling Game

We can turn anything into a game!

Two players take turns tiling a board using 2x1 dominoes. Whichever cannot place a new domino loses. If you cover the entire board, it's a draw!

Find yourself a partner and play on the following boards. Is there always an optimal strategy for one of the players? Can you tell someone always wins, or if it's always a draw?

ANALYSIS FOR BOARD 2

There are 3 possible first moves (up to reflection/rotation symmetry) for **Player 1** to split the board.

The other moves for **Player 2** have all a piece of the board with 1 or 2 squares.

Can Player 1 still prevent Player 2 from forcing off another square? (find a move for Player 1)



Thank you for your time!

Miro Tutorial



GeoGebra Tutorial



Please do not hesitate to reach out to us!

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